The Cylindrical Antenna with Nonreflecting Resistive Loading

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Abstract—**The distribution of current along a center-driven cylindrical antenna is obtained when the material forming the antenna is resistive. The particular case is considered when the impedance per unit length of the antenna is a function of the distance from the end. A solution is obtained specifically when the current is represented by an outward traveling wave with no reflected wave. The admittance of the antenna and the far-field pattern is determined. Field patterns are evaluated for a wide range of lengths. These are characterized by a single major lobe with a very small minor lobe structure.**

Introduction

FOR SOME PURPOSES, the directional and broadband properties of traveling-wave antennas are desirable. An example is the traveling-wave V-antenna. The first work on traveling-wave dipoles was reported by Altschuler[1] who inserted lumped resistors at a quarter wavelength from the ends of the antenna. Although this location of the resistors is not critical, the traveling-wave nature of the current diminishes as the frequency is changed so that the lumped resistors are no longer at the maximum of the current.

In a recent report[2] the distribution of current and the driving-point admittance were determined for a cylindrical antenna with a continuously distributed constant internal impedance per unit length. It is now proposed to investigate the cylindrical antenna with a variable internal impedance per unit length. In particular, it is desired to determine an axial distribution of the internal impedance for which a pure outward traveling wave exists on an antenna of finite length.

The Differential Equation and Its Solution

The axial component $A_z(z)$ of the vector potential on the surface of a cylindrical antenna, that has the internal impedance per unit length $z^i(z)$, carries a total axial current $I_z(z)$, and is driven at $z=0$ by a delta-function generator with emf V_0^e, satisfies the one-dimensional wave equation in the form

$$\left(\frac{\partial^2}{\partial x^2}+k_0^2\right)A_z(z)=\frac{jk_0^2}{\omega}\left[z^i(z)I_z(z)-V_0^e\delta(z)\right] \tag{1}$$

if a time-dependence $e^{j\omega t}$ is assumed. The internal impedance per unit length $z^i(z)$ is expressed as a function of the axial coordinate z. It is given by

$$z^i(z)=\frac{1}{2\pi a d(z)\sigma(z)} \tag{2}$$

for a circular tube with constant radius a. In order to vary the impedance per unit length, it is assumed that the conductivity σ and the wall thickness d may be functions of location along the antenna. The vector potential on the surface of the antenna is

$$A_z(z)=\frac{\mu_0}{4\pi}\int_{-h}^{h}I_z(z')K(z,z')dz' \tag{3}$$

where

$$K(z,z')=\frac{e^{-jk_0r}}{r} \tag{4}$$

with

$$r=\sqrt{(z-z')^2+a^2} \tag{5}$$

Since the ratio of vector potential to current along an antenna is approximately constant, it is possible to set

$$\int_{-h}^{h}I_z(z')K(z,z')dz'\doteq I_z(z)\Psi \tag{6}$$

where Ψ is the value where the current $I_z(z)$ has a maximum.

With (3)–(6) it follows that

$$\left(\frac{\partial^2}{\partial z^2}+k_0^2\right)4\pi\mu_0^{-1}A_z(z)\doteq\frac{j4\pi k_0}{\zeta_0}\left[z^i(z)I_z(z)-V_0^e\delta(z)\right] \tag{7}$$

may be approximated by

$$\left(\frac{\partial^2}{\partial z^2}+k_0^2\right)I_z(z)=\frac{j4\pi k_0}{\zeta_0\Psi}\left[z^i(z)I_z(z)-V_0^e\delta(z)\right] \tag{8}$$

With the notation

$$f(z)=\frac{4\pi}{\zeta_0\Psi}z^i(z) \tag{9}$$

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[1] Altshuler, E. E., The traveling-wave linear antenna, *IRE Trans. on Antennas and Propagation*, vol AP-9, Jul 1961, pp 324–329.

[2] King, R. W. P., and W. T. Tsun, The imperfectly conducting cylindrical transmitting antenna, Tech Rept 440, Cruft Lab., Harvard University, Cambridge, Mass., Mar 1964.

where $\zeta_0=\sqrt{\mu_0/\epsilon_0}=120\pi$ ohms, this equation becomes

$$\left[\frac{\partial^2}{\partial z^2}+k_0^2-jk_0f(z)\right]I(z)=-\frac{j4\pi k_0}{\zeta_0\Psi}V_0^e\delta(z) \qquad (10)$$

Except at the driving point $z=0$, the current must satisfy the differential equation

$$\left[\frac{\partial^2}{\partial z^2}+k_0^2-jk_0f(z)\right]I_z(z)=0 \qquad (11)$$

It is readily verified by direct substitution in (11) that when

$$f(z)=\frac{2}{h-|z|} \qquad (12)$$

so that (11) becomes

$$\left(\frac{\partial^2}{\partial z^2}+k_0^2-\frac{j2k_0}{h-|z|}\right)I_z(z)=0 \qquad (13)$$

a solution is

$$I_z(z)=C(h-|z|)e^{-jk_0|z|} \qquad (14)$$

Note, in particular, that a solution of the form $e^{jk|z|}$ does not satisfy the equation.

The expression (14) represents a wave of current traveling in the direction of increasing $|z|$, that is, from the generator toward both ends. There is no reflected wave traveling in the opposite direction.

The Impedance and Expansion Parameter

If the current has the form (14), it follows that the vector potential is given by

$$4\pi\mu_0^{-1}A_z(z)=\Psi I_z(z)=\Psi C(h-|z|)e^{-jk_0|z|} \qquad (15)$$

The scalar potential satisfies the Lorentz condition

$$\phi(z)=j\frac{\omega}{k_0^2}\frac{\partial A_z(z)}{\partial z} \qquad (16)$$

Also, by symmetry, $\phi(-z)=-\phi(z)$. For $z\geq 0$

$$\phi(z)=\frac{j\omega\mu_0}{4\pi k_0^2}\Psi Ce^{-jkz}[-1-jk_0(h-z)] \qquad (17)$$

$$\phi(+0)=\frac{j\omega\mu_0}{4\pi k_0^2}\Psi C(1+jk_0h)=\frac{-j}{4\pi\omega\epsilon_0}\Psi C(1+jk_0h) \qquad (18)$$

If the driving voltage is defined by

$$V_0^e=\phi(+0)-\phi(-0)=2\phi(+0) \qquad (19)$$

it follows that

$$C=\frac{j2\pi\omega\epsilon_0V_0^e}{\Psi(1+jk_0h)} \qquad (20)$$

Hence,

$$I(z)=\frac{2\pi V_0^e}{\zeta_0\Psi(1-j/k_0h)}\left(1-\frac{|z|}{h}\right)e^{-jk_0|z|} \qquad (21)$$

The driving-point admittance is

$$Y_0=\frac{2\pi}{\zeta_0\Psi}\frac{1}{1-j/k_0h} \qquad (22)$$

The impedance is a resistance in series with a capacitance

$$Z_0=R_0-j/\omega C_0 \qquad (23)$$

where

$$R_0=\frac{\Psi\zeta_0}{2\pi}=60\Psi \text{ ohms} \qquad (24)$$

and

$$C_0=\epsilon_0h \qquad (25)$$

Note that when $k_0h\gg 1$, $R_0\gg 1/\omega C_0$.

The parameter Ψ is defined in terms of the function

$$\Psi(z)=\frac{\int_0^h(h-z')e^{-jkz'}\left[\frac{e^{-jkr_1}}{r_1}+\frac{e^{-jkr_2}}{r_2}\right]dz'}{(h-z)e^{-jkz}} \qquad (26)$$

where

$$r_1=\sqrt{(z'-z)^2+a^2} \qquad r_2=\sqrt{(z'+z)^2+a^2}$$

Since $I(z)$ and $A_z(z)$ both have maximum amplitudes at $z=0$, it is desirable to define $\Psi=\Psi(0)$. That is,

$$\Psi=2\int_0^h\left(1-\frac{z'}{h}\right)e^{-jkz'}\frac{e^{-jkr_0}}{r_0}dz' \qquad (27)$$

where

$$r_0=\sqrt{z'^2+a^2}$$

Since $k_0a\ll 1$, and $a\ll h$, no serious error is made by setting $kz'\doteq kr_0$ in the exponent in the first integral and $r_0\doteq z'$ in the second integral.

$$\Psi\doteq 2\int_0^h\frac{e^{-j2kr_0}}{r_0}dz'-\frac{2}{h}\int_0^h e^{-j2kz'}dz' \qquad (28)$$

With $A=ka$, it follows that

$$\Psi\doteq 2\left[\sinh^{-1}\frac{h}{a}-C(2A,2kh)-jS(2A,2kh)\right]+\frac{1}{kh}(1-e^{-j2kh}) \qquad (29)$$

Specifically, when $kh=\pi/2$, $h/a=75$, $\Omega=10$, $ka=1.57/75=0.021$, $2ka=0.042$,

$$\Psi \doteq 2[5.70 - 1.66 - j1.85] + j\frac{2}{\pi}(1+1)$$

$$= 8.08 - j2.43$$

Similarly, for a thin antenna with $h/a=11{,}013$, or $\Omega=20$, $ka=1.57/11,013=1.41\times10^{-4}$

$$\Psi \doteq 2[10.69 - 1.65 - j1.85] + j\frac{4}{\pi} = 18.08 - j2.43$$

In (29), $C(a, x)$ and $S(a, x)$ are the generalized sine and cosine integrals:

$$C(a, x) = \int_0^x \frac{1-\cos W}{W}du \qquad S(a, x) = \int_0^x \frac{\sin W}{W}du$$

where

$$W = (u^2 + a^2)^{1/2}$$

The Distributed Resistive Loading

The continuously varying resistive loading of the antenna is defined by (9) with (12). Thus

$$z^i(z) = \frac{\zeta_0\Psi}{8\pi}\frac{1}{h-|z|} = \frac{15\Psi}{h-|z|} \tag{30}$$

where the coefficient 15 is in ohms. With $\Omega=10$ to 20, the coefficient 15Ψ ohms ranges from $121-j36$ ohms to $272-j36$ ohms. At $\lambda=288$ m with $\Omega=20$, $h=72m$ so that

$$z^i(0) = 15\Psi/h = \frac{272-j36}{72} = 3.9 - j.5 \text{ ohms}/m$$

is the impedance per unit length at the driving point. In this case $a=6.54$ mm. With $z^i=\frac{1}{2}\pi a d\sigma$, $z_i=0.66$, 6.6, and 66 ohms/m for aluminum of thickness $d=10^{-6}$, 10^{-7}, and 10^{-8} m and $z^i=8.55$, 85.5, and 855 ohms/m for carbon of thickness $d=10^{-4}$, 10^{-5}, and 10^{-6} m. Thus, with thin layers of aluminum or carbon, a range from below 3.9 ohms/m at $z=0$ to very large values as $z\rightarrow h$ may be constructed.

Efficiency

The efficiency of the resistively loaded antenna is easily evaluated since

$$P_t = \frac{1}{2}|I(0)|^2R_0 = \frac{\zeta_0\Psi}{4\pi}|I(0)|^2$$

whereas the power dissipated in heat is

$$P_h = 2\int_0^h \frac{1}{2}|I(z)|^2 z^i(z)dz = \frac{\zeta_0\Psi}{8\pi}|I(0)|^2.$$

It follows that the antenna is 50 per cent efficient.

The Electromagnetic Field

The far-zone electric field is given by

$$E_\Theta^r = j\omega\sin\Theta\, A_z^r \tag{31}$$

where

$$A_z^r = \frac{e^{-jk_0r_0}}{4\pi\nu_0r_0}\int_{-h}^h I_z(z')e^{jk_0z'\cos\Theta^1}dz' \tag{32}$$

With

$$I(z') = C(h - |z|)e^{-jk_0|z|} \tag{33}$$

this expression becomes

$$A_z^r = \frac{Ce^{-jk_0r_0}}{2\pi\nu_0r_0}\int_0^h (h-z')e^{-jk_0z'}\cos(k_0z'\cos\Theta)dz' \tag{34}$$

The integral is readily evaluated with the following formulas:

$$\int e^{ax}\cos bx dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) \tag{35a}$$

$$\int xe^{ax}\cos bxdx = \frac{e^{ax}}{a^2+b^2}\left\{\left[ax - \frac{a^2-b^2}{a^2+b^2}\right]\cos bx + \left[bx - \frac{2ab}{a^2+b^2}\right]\sin bx\right\} \tag{35b}$$

Thus,

$$\int_0^h (h-z')e^{az'}\cos bz'dz'$$

$$= \frac{e^{az'}}{a^2+b^2}\left\{\left[a(h-z') + \frac{a^2-b^2}{a^2+b^2}\right]\cos bz' + \left[b(h-z') + \frac{2ab}{a^2+b^2}\right]\sin bz'\right\}$$

$$= \frac{1}{(a^2+b^2)^2}\{[(a^2-b^2)\cos bh + 2ab\sin bh]e^{ah} - [ah(a^2+b^2) + a^2 - b^2]\} \tag{36}$$

With $a=-jk_0$, $b=k_0\cos\Theta$, $a^2+b^2=-k_0^2\sin^2\Theta$, $a^2-b^2=-k^2(1+\cos^2\Theta)$, it follows that

$$\int_0^h (h-z')e^{-jk_0z'}\cos(k_0z'\cos\Theta)dz'$$

$$= \frac{[-(1+\cos^2\Theta)\cos(k_0h\cos\Theta) - j2\cos\Theta\sin(k_0h\cos\Theta)]e^{-jk_0h} - jk_0h\sin^2\Theta + (1+\cos^2\Theta)}{k_0^2\sin^4\Theta} \tag{37}$$

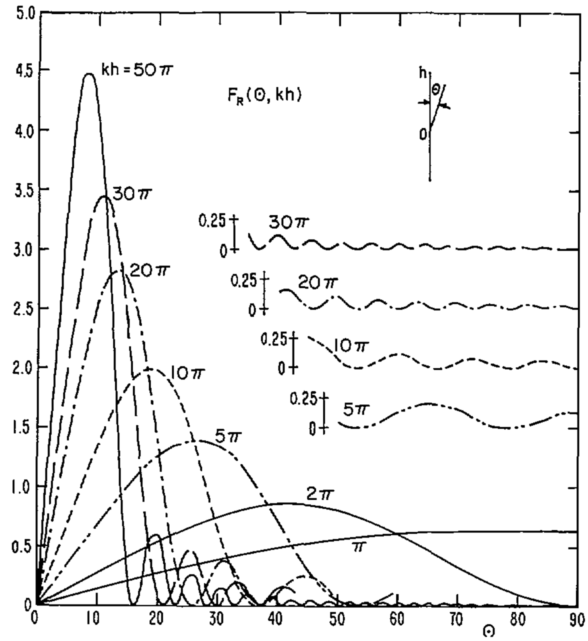

Fig. 1. Real part of far-field pattern of nonreflecting antenna.

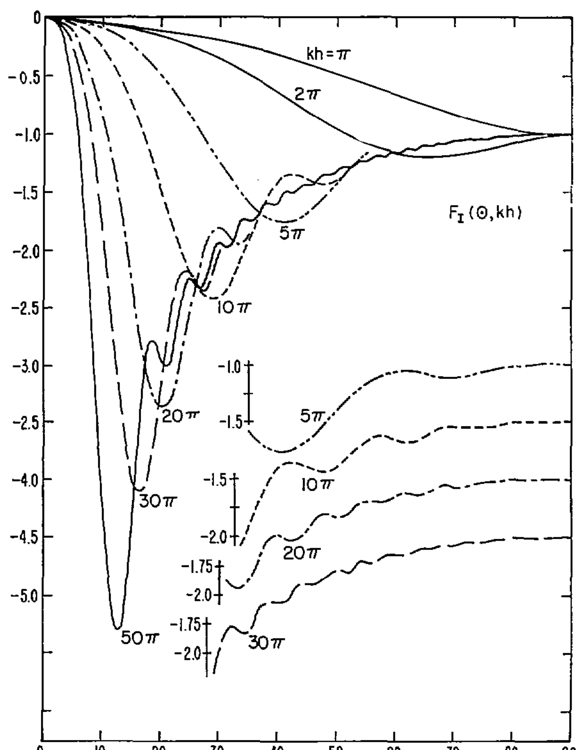

Fig. 2. Imaginary part of far-field pattern of nonreflecting antenna.

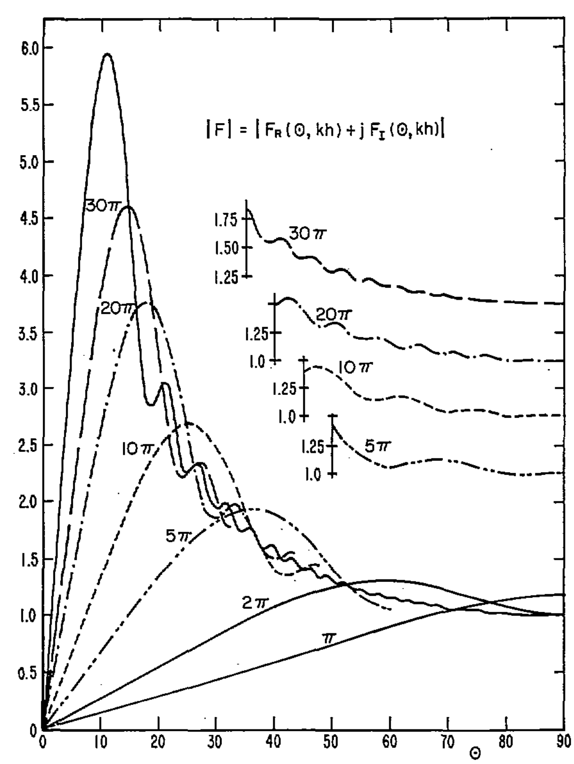

Fig. 3. Magnitude of far-field pattern of nonreflecting antenna.

The electric field is

$$E_{\Theta}^{r} = \frac{j\zeta_0 h C e^{-jk_0 r}}{2\pi r} F(k_0 h, \Theta) \tag{38a}$$

where the vertical field factor is

$$F(k_0 h, \Theta) = \frac{-jk_0 h \sin^2\Theta + (1+\cos^2\Theta) - [j2\cos\Theta \sin(k_0 h \cos\Theta) + (1+\cos^2\Theta)\cos(k_0 h\cos\Theta)]e^{-jk_0 h}}{k_0 h \sin^3\Theta} \tag{38b}$$

This function vanishes along the axis $\Theta = 0$ and has the value

$$F\left(k_0 h, \frac{\pi}{2}\right) = \frac{-j(k_0 h - \sin k_0 h) + (1 - \cos k_0 h)}{k_0 h} \tag{39}$$

in the equatorial plane, $\Theta = \pi/2$. When $k_0^2 h^2 \ll 1$,

$$F(k_0 h, \Theta) = k_0 h \sin\Theta \tag{40}$$

which is the same as for any electrically short antenna. The real and imaginary parts are

$$F_R(k_0 h, \Theta) = \frac{(1+\cos^2\Theta)[1 - \cos k_0 h \cos(k_0 h \cos\Theta)] - 2\cos\Theta \sin k_0 h \sin(k_0 h\cos\Theta)}{k_0 h \sin^3\Theta} \tag{41a}$$

$$F_I(k_0 h, \Theta) = \frac{-k_0 h \sin^2\Theta - 2\cos\Theta\cos k_0 h \sin(k_0 h\cos\Theta) + (1+\cos^2\Theta)\sin k_0 h \cos(k_0 h \cos\Theta)}{k_0 h \sin^3\Theta} \tag{41b}$$

When

$$k_0 h = \frac{\pi}{2}$$

$$F_R\left(\frac{\pi}{2}, \Theta\right) = \frac{1 + \cos^2\Theta - 2\cos\Theta \sin\left(\frac{\pi}{2}\cos\Theta\right)}{\frac{\pi}{2}\sin^3\Theta} \tag{42a}$$

$$F_I\left(\frac{\pi}{2}, \Theta\right) = \frac{-\frac{\pi}{2}\sin^2\Theta + (1+\cos^2\Theta)\cos\left(\frac{\pi}{2}\cos\Theta\right)}{\frac{\pi}{2}\sin^3\Theta} \tag{42b}$$

At

$$\Theta = \frac{\pi}{2}, \qquad F_R\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = \frac{2}{\pi}, \qquad F_I\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = \frac{2}{\pi} - 1.$$

When $k_0 h \gg 1$ $F(k_0 h, \Theta) \to F_I(k_0 h, \Theta) \to -\csc\Theta$, $\Theta \neq 0$.

Graphs of $F_R(k_0 h, \Theta)$, $F_I(k_0 h, \Theta)$, and $|F(k_0 h, \Theta)|$ are given in Figs. 1–3 for a range of values of $k_0 h$ extending from $\pi/2$ to 50π. It is seen that $|F(k_0 h, \Theta)|$ has one large maximum that is located at $\Theta = 90°$ for $k_0 h \leq \pi$ and moves toward $\Theta = 0$ as $k_0 h$ is increased. With $k_0 h = 50\pi$ the maximum is near $\Theta = 11°$. Of particular interest is the fact that minor lobes are little more than a small ripple on the broad tail of the major maximum for all values of $k_0 h > \pi$.

Conclusion

Properties of a center-driven cylindrical antenna, characterized by a pure traveling wave of current, was investigated. Combination of two such antennas into traveling-wave V-antennas will be treated in another paper.